| RGC Ref.: N_CUHK437/16<br>NSFC Ref. : 11661161017 |
| --- |
| *(please insert ref. above)* |

**The Research Grants Council of Hong Kong**
**NSFC/RGC Joint Research Scheme**
**Joint Completion Report**

*(Please attach a copy of the completion report submitted to the NSFC by the Mainland researcher)*

## Part A: The Project and Investigator(s)

**1. Project Title**
**Mathematical and numerical study of non-conforming finite element methods for Maxwell's equations in inhomogeneous media and related inverse problems**

**2. Investigator(s) and Academic Department/Units Involved**

| | Hong Kong Team | Mainland Team |
| --- | --- | --- |
| Name of Principal Investigator *(with title)* | Zou Jun | Duan Huoyuan |
| Post | Choh-Ming Li Professor of Mathematics,<br>and Chairman,<br>Department Mathematics | Professor of Mathematics |
| Unit / Department / Institution | Department Mathematics, The Chinese University of Hong Kong | School of Mathematics and Statistics, Wuhan University |
| Contact Information | zou@math.cuhk.edu.hk | hyduan.math@whu.edu.cn |
| Co-investigator(s) *(with title and institution)* | N/A | Jiang Daijun,<br>Associate Professor,<br>School of Mathematics and Statistics,<br>Central China Normal University;<br>Xiang Hua, Professor,<br>School of Mathematics and Statistics, Wuhan University |

## 3. Project Duration

| | Original | Revised | Date of RGC/ Institution Approval (*must be quoted*) |
|---|---|---|---|
| Project Start date | 1 January 2017 | 1 January 2017 | |
| Project Completion date | 31 December 2020 | 30 June 2021 | 1 June 2020 |
| Duration *(in month)* | 48 | 54 | 1 June 2020 |
| Deadline for Submission of Completion Report | 31 December 2021 | 30 June 2022 | 1 June 2020 |

## Part B: The Completion Report

### 5. Project Objectives

5.1 Objectives as per original application

1. To develop non-conforming finite element methods (NFEMs) for three-dimensional Maxwell's equations in inhomogeneous media with non-$H^1$ or high frequency solutions.

2. To conduct a large number of numerical experiments for the newly proposed NFEMs, and improve the NFEMs based on the numerical experiments so that the resulting NFEMs achieve optimal convergence for Maxwell's equations in inhomogeneous media.

3. To establish the stability, convergence and error estimates of the new non-conforming finite element methods.

4. To apply the proposed NFEMs to solve some other important mathematical and physical model equations that involve the div or curl operator or both.

5. To study the mathematical formulation and regularizations for the inverse problem of recovering the electric permittivity and magnetic permeability in Maxwell's equations in inhomogeneous media, and analyze the stability and convergence of the discretized regularizations of the Maxwell inverse problem by the new NFEMs.

5.2 Revised Objectives

The same as the original objectives.

## 6. Research Outcome

Major findings and research outcome *(maximum 1 page; please make reference to Part C where necessary)*

Due to one-page limitation, we describe those most important findings and research outcome below.

We have developed a series of efficient numerical methods and the fast numerical algorithms and solvers based on these methods for various Maxwell related model systems in inhomogeneous media with singular or high oscillating solutions. These include optimal controls of Maxwell system [1], the linear and nonlinear acoustic and optical wave systems [3][6], electromagnetic scattering problems [5], time-harmonic Maxwell systems [11], the eddy current problem [12] and Maxwell variational inequalities [14]. In addition, we have also developed the adaptive FEMs for the quasilinear Maxwell system [15] and a group of mixed FEMs for the Maxwell eigenproblem [16]. We have conducted a very large number of numerical experiments to demonstrate and verify the efficiency, stability and accuracies of all these newly proposed numerical methods and the fast numerical algorithms and solvers.

We have developed a series of systematical frameworks and theories that enabled us to establish the stability, convergence and error estimates for all the new numerical methods and the fast numerical algorithms initiated and proposed in this project for various Maxwell related model systems in inhomogeneous media with singular or high oscillating solutions as listed in the above publications [1][3][5][6][11][12][14][15]16].

We have applied our newly developed numerical methods in [1][3][5][6][11][12][14][15]16] to some other important mathematical and physical models of partial differential equations. This includes the magnetohydrodynamic flow [2], the singular Maxwell saddle point systems [4], the eddy current Maxwell problem [12], the nonlinear fractional systems [13] and the Maxwell variational inequality system [14] and the quasilinear Maxwell system [15].

We have developed in this project a series of mathematical and numerical analysis theories for solving various inverse problems that are closely related to Maxwell systems in inhomogeneous media. This includes a new theory that enables us to establish the unique identifiability of the shape and physical properties of the inhomogeneous inclusions associated with both the inverse acoustic and Maxwell scattering problems [7][8]; we developed some creative theories for analysing electromagnetic plasmonic metasurfaces and mathematically explain essential physical changes of electromagnetic metasurface at resonances [9], as well as for quantitatively understanding the superresolution in recovering embedded electromagnetic sources in high contrast media [10]; we conducted a systematic mathematical and numerical study of an inverse eddy current Maxwell model in inhomogeneous media, formulated the inverse problem as a constrained optimization with a Tikhonov regularization, and analysed the stability and convergence of the discretized regularized FE solutions [12]; we developed new theories to enable us to achieve convergence rates of Tikhonov regularizations for elliptic and parabolic inverse radiativity problems [17] and the inverse Robin and flux problems by partial measurements [18], and the quadratic convergence of Levenberg-Marquardt method for elliptic and parabolic inverse Robin problems [19].

In addition to many other joint collaboration outcome, the major findings and research outcome reported in publications [11][16][17][18][19] were the most important results achieved by the joint collaboration between the Hong Kong and Mainland teams.

Potential for further development of the research and the proposed course of action
*(maximum half a page)*

Motived by the theoretical and numerical results obtained in this project, we find the following topics of great interest and potential that deserve further development.

The numerical solutions of the large-scale finite element systems arising from various Maxwell's systems in inhomogeneous media are mostly computationally very expensive. It has remained to be a great challenge to construct efficient solvers for these large-scale systems. One of the most promising strategies for this is to develop efficient preconditioning type iterative solvers for solving such large-scale discrete systems so that the convergence and convergence rates of these solvers are optimal. And more importantly, it is of great desire to construct iterative solvers whose convergence is independent of some key parameters, such as the physical parameters in the model systems, the mesh size, the regularization parameters and high wavenumbers. We have made some important progress in this direction through this project and developed efficient preconditioning type iterative solvers for some special Maxwell systems. But tremendous more effort should be made in this direction.

In addition, to better understand the aforementioned efficient preconditioning type iterative solvers and their performance, it is necessary and of great practice to develop some general frameworks and theories for analysing the efficiency, stability and complexities of these iterative solvers.

## 7. The Layman's Summary

*(describe <u>in layman's language</u> the nature, significance and value of the research project, in no more than 200 words)*

We have developed a series of novel numerical methods for more efficiently solving various Maxwell's systems in inhomogeneous media and their related inverse problems. We have also established a set of systematic mathematical theories to help analyse and understand the stability and convergence of these numerical methods, especially for the challenging cases with high frequency solutions, non-solutions, or solutions with very low regularities in the entire physical domain. The numerical solutions of the large-scale discretized systems arising from various Maxwell's systems in inhomogeneous media have been of great challenge. This project has made a significant contribution to this challenging and frontier area, and developed some efficient solvers for these large-scale discrete systems and proved that the convergence and convergence rates of these solvers are optimal, independent of the mesh size and wavenumbers.

The numerical methods and solvers that are developed in this project are efficient and effective not only for the Maxwell's systems, but also for many other important mathematical models of partial differential equations. These new numerical methods are also applied for solving the related mathematically ill-posed Maxwell inverse problems, and their stability and convergence can be analysed using the mathematical theories established in this project.

## Part C: Research Output

### 8. Peer-reviewed journal publication(s) arising directly from this research project

*(Please attach a copy of each publication and/or the letter of acceptance if not yet submitted in the previous progress report(s). All listed publications must acknowledge RGC's funding support by quoting the specific grant reference.)*

| The Latest Status of Publications | | | | Author(s) (**bold** the authors belonging to the project teams and denote the corresponding author with an asterisk*) | Title and Journal/ Book (with the volume, pages and other necessary publishing details specified) | Submitted to RGC (indicate the year ending of the relevant progress report) | Attached to this report (Yes or No) | Acknowledged the support of this Joint Research Scheme (Yes or No) | Accessible from the institutional repository (Yes or No) |
|---|---|---|---|---|---|---|---|---|---|
| Year of publication | Year of Acceptance (For paper accepted but not yet published) | Under Review | Under Preparation (optional) | | | | | | |
| (1) 2017 | | | | Irwin Yousept and **Jun Zou*** | Edge Element method for Optimal control of stationary Maxwell system with Gauss Law. SIAM J. Numer. Anal., 55 (2017), 2787–2810. | 31-12-2018 | Yes | Yes | Yes |
| (2) 2018 | | | | Yinnian He and **Jun Zou*** | A priori estimates and optimal finite element approximation of the MHD flow in smooth domains, ESAIM: Math. Model. Numer. Anal. 52 (2018) 181-206. | 31-12-2018 | Yes | Yes | Yes |
| (3) 2018 | | | | Haijun Wu and **Jun Zou***, | Finite element method and its analysis for a nonlinear Helmholtz equation with high wave numbers. SIAM J. Numer. Anal. 56 (2018), 1338-1359. | 31-12-2018 | Yes | Yes | Yes |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (4) 2018 | | | | Na Huang, Chang-Feng Ma and **Jun Zou***, | Spectral Analysis, Properties and Nonsingular Preconditioners for Singular Saddle Point Problems, Comput. Methods Appl. Math. 18 (2018), 237–256. | 31-12-2018 | Yes | Yes | Yes |
| (5) 2017 | | | | Na Huang, Chang-Feng Ma and **Jun Zou***, | Analysis on block diagonal and triangular preconditioners for a PML system of an electromagnetic scattering problem. Computers and Mathematics with Applications 74 (2017), 2856–2873. | 31-12-2018 | Yes | Yes | Yes |
| (6) 2020 | | | | I.G. Graham, E.A. Spence and **Jun Zou***, | Domain decomposition with local impedance conditions for the Helmholtz equation with absorption. SIAM J. Numer. Anal. 58 (2020), 2515-2543. | 31-12-2018 | Yes | Yes | Yes |
| (7) 2021 | | | | Huaian Diao, Long Zhang, Hongyu Liu and **Jun Zou***, | Unique continuation from a generalized impedance edge-corner for Maxwell's system and applications to inverse problems. Inverse Problems 37 (2021), 035004. | | Yes | Yes | Yes |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (8) 2021 | | | | Xinlin Cao, Huaian Diao, Hongyu Liu and **Jun Zou***, | On novel geometric structures of Laplacian eigenfunctions in $R^3$ and applications to inverse problems. SIAM J. Math. Anal. 53 (2021), 1263-1294. | | Yes | Yes | Yes |
| (9) 2020 | | | | Habib Ammari, Bowen Li and **Jun Zou***, | Mathematical analysis of electromagnetic plasmonic metasurfaces. SIAM Multiscale Model. Simul. 18 (2020), 758-797. | | Yes | Yes | Yes |
| (10) 2020 | | | | Habib Ammari, Bowen Li and **Jun Zou***, | Superresolution in recovering embedded electromagnetic sources in high contrast media. SIAM J. Imag. Sci. 13 (2020), 1467-1510. | | Yes | Yes | Yes |
| (11) 2021 | | | | Ying Liang, Hua Xiang, Shiyang Zhang and **Jun Zou***, | Preconditioners and their analyses for edge element saddle-point systems arising from time-harmonic Maxwell equations. Numerical Algorithms 86 (2021), 281-302. | | Yes | Yes | Yes |
| (12) 2020 | | | | Junqing Chen, Ying Liang and **Jun Zou***, | Mathematical and numerical study of a three-dimensional inverse eddy current problem. SIAM J. Appl. Math. 80 (2020), 1467-1492. | | Yes | Yes | Yes |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (13) 2020 | | | | Dongling Wang, Aiguo Xiao and **Jun Zou***, | Long-time behavior of numerical solutions to nonlinear fractional ODEs. ESAIM: Math. Model. Numer. Anal. (M2AN) 54 (2020), 335-358. | | Yes | Yes | Yes |
| (14) 2020 | | | | Malte Winckler, Irwin Yousept and **Jun Zou***, | Adaptive edge element approximation for H(curl) elliptic variational inequalities of second kind. SIAM J. Numer. Anal. 58 (2020), 1941-1964. | | Yes | Yes | Yes |
| (15) 2020 | | | | Yifeng Xu, Irwin Yousept and **Jun Zou***, | An adaptive edge element approximation of a quasilinear H(curl)-elliptic problem. Math. Models Methods Appl. Sci. (M3AS) 30 (2020), 2799-2826. | | Yes | Yes | Yes |
| (16) 2021 | | | | Huoyuan Duan, Junhua Ma and **Jun Zou***, | Mixed finite element method with Gauss's law enforced for the Maxwell eigenproblem. SISC J Sci. Comput. 43 (2021), A3677-A3712. | | Yes | Yes | Yes |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (17) 2020 | | | | De-Han Chen, Daijun Jiang and **Jun Zou***, | Convergence rates of Tikhonov regularizations for elliptic and parabolic inverse radiativity problems. Inverse Problems 36 (2020), 075001. | | Yes | Yes | Yes |
| (18) | 2021 | | | De-Han Chen, Daijun Jiang, Irwin Yousept and **Jun Zou***, | Variational source conditions for inverse Robin and flux problems by partial measurements. Inverse Problems and Imaging, accepted in 2021. doi: 10.3934/ipi.2021050. | | Yes | Yes | Yes |
| (19) 2018 | | | | Daijun Jiang, Hui Feng and **Jun Zou***, | Quadratic convergence of Levenberg-Marquardt method for elliptic and parabolic inverse Robin problems. ESAIM: Math. Model. Numer. Anal. (M2AN) 52 (2018), 1085-1107. | | Yes | Yes | Yes |

**9. Recognized international conference(s) in which paper(s) related to this research project was/were delivered** *(Please attach a copy of each delivered paper. All listed papers must acknowledge RGC's funding support by quoting the specific grant reference.)*

| Month/Year/ Place | Title | Conference Name | Submitted to RGC *(indicate the year ending of the relevant progress report)* | Attached to this report *(Yes or No)* | Acknowledged the support of this Joint Research Scheme *(Yes or No)* | Accessible from the institutional repository *(Yes or No)* |
|---|---|---|---|---|---|---|
| May 29 - June 2, 2017, Hangzhou, China. | Adaptive finite element methods for general inverse problems | International Applied Inverse Problems Conference | 31-12-2018 | Yes | Yes | No |
| September 29-October 2, 2018, Yonsei, Korea | Direct Sampling Methods for Nonlinear Inverse Problems | International Conference on Computational Mathematics | 31-12-2018 | Yes | Yes | No |
| 09/2018/ Singapore | Efficient adaptive nodal finite element methods for general ill-posed inverse problems. | Workshop on Qualitative and Quantitative Approaches to Inverse Scattering Problems, September 24-28, 2018, Singapore. | | Yes | Yes | No |
| 03/2019/ Spokane, USA | Direct sampling methods for general inverse problems. | SIAM Conference on Computational Science and Engineering (CSE19), February 25-March 1, 2019 at the Spokane Convention Center, Spokane, Washington, USA. | | Yes | Yes | No |

| | | | | | | |
|---|---|---|---|---|---|---|
| 06/2021/ Changsha, China | Robust and fast direct sampling methods for nonlinear inverse problems. | Workshop on Efficient and fast numerical methods and applications. | | Yes | Yes | No |
| 06/2021/ Chengdu, China | Robust Preconditioners for Large-scale Systems Arising from Various Maxwell Equations. | Symposium on Scientific Computing: Theory and Applications. | | Yes | Yes | No |

## 10. Student(s) trained *(Please attach a copy of the title page of the thesis.)*

| Name | Degree registered for | Date of registration | Date of thesis submission/ graduation |
|---|---|---|---|
| Qile Yan | MPhil | August 2016 | August 2018 |
| Xin Su | MPhil | August 2016 | July 2018 |
| Ying Liang | PhD | August 2016 | May 2021 |
| Bowen Li | PhD | August 2017 | May 2021 |

## 11. Other impact *(e.g. award of patents or prizes, collaboration with other research institutions, technology transfer, etc.)*

**Honours/prizes Hong Kong PI received:**
the PI was elected as a Fellow of the Society for Industrial and Applied Mathematics (SIAM) in 2019;
the PI was elected as a Fellow of the American Mathematical Society (AMS) in 2022.
SIAM and AMS are the most largest and influential mathematical societies in the world. SIAM Fellows and AMS Fellows are the highest academic honours a mathematician can receive from the two societies.
the PI received the prestigious China National Thousand Talents Program through Wuhan University.

**Collaborations:**
With the fundamental support of this joint NSFC/RGC research project, along with partial support from Chinese University of Hong Kong, the PI has taken part in many international conferences and delivered invited talks, and visited the following leading researchers who work in the areas that are closely related to the project. The PI has also invited some of the leading experts in the related areas to visit the PI and his research team, as specified below. The PI has established close and successful collaborations with most of these experts; see also detailed research activities and outcomes in Item 6 that were carried out and achieved during this project. Most of these visits by PI or his collaborators could not continue after the outbreak of the worldwide COVID-19 pandemic since the start of 2020.

Visits by collaborators at CUHK:

Prof. Bernd Hofmann (Technical University of Chemnitz, Germany).
(2 weeks, 2017)

Prof. Kazufumi Ito (North Carolina State University, USA).
(4 weeks 2017; 4 weeks, 2018; 4 weeks, 2019)

Prof. Keke Zhang (Exeter University, UK; 2 weeks: 2017; 2 weeks: 2018).

Dr. Yikan Liu (University of Yokyo, Japan: 4 weeks, 2018).

Prof. Patrick Ciarlet, Jr. (ENSTA, France: one week: 2018)

Prof. Bangti Jin (University of College London, UK;
3 weeks, 2017; 4 weeks, 2018; 4 weeks: 2019; 2 months: 2020; 4 months: 2021)

Prof. Iva Graham (Bath University, UK:
3 months, 2017; 3 weeks, 2018; 2 weeks, 2019)

Prof. Huoyuan Duan (Wuhan University, China: 3 weeks, 2017; 2 months, 2018)

Prof. Zhiming Chen
(Chinese Academy of Sciences, China: 2 months, 2018; one week, 2019)

**12. Statistics on Research Outputs** *(Please ensure the summary statistics below are consistent with the information presented in other parts of this report.)*

| | Peer-reviewed journal publications | Conference papers | Scholarly books, monographs and chapters | Patents awarded | Other research outputs (Please specify) |
|---|---|---|---|---|---|
| No. of outputs arising directly from this research project [or conference] | 19 | 6 | | | |