RGC Ref. No.:
UGC/FDS14/E01/14
(please insert ref. above)

RESEARCH GRANTS COUNCIL
COMPETITIVE RESEARCH FUNDING SCHEMES FOR THE LOCAL SELF-FINANCING DEGREE SECTOR

FACULTY DEVELOPMENT SCHEME (FDS)

Completion Report

(for completed projects only)

Submission Deadlines:	*1. Auditor's report with unspent balance, if any: within **six** months of the approved project completion date.* *2. Completion report: within **12** months of the approved project completion date.*

Part A: The Project and Investigator(s)

1. Project Title

On 2-d rectangular packing problem with aspect ratio considerations

2. Investigator(s) And Academic Department(s) / Unit(s) Involved

Research Team	Name / Post	Unit / Department / Institution
Principal Investigator	Chan Chi Kong / Lecturer I	Dept of Computing, Hang Seng Management College
Co-Investigator(s)	Chan Tsz Lun / Lecturer I	Dept of Computing, Hang Seng Management College
Others		

3. Project Duration

	Original	Revised	Date of RGC / Institution Approval *(must be quoted)*
Project Start Date	01/01/2015	NA	NA
Project Completion Date	31/12/2015	30/6/2016	22/5/2015
Duration *(in month)*	12	18	22/5/2015
Deadline for Submission of Completion Report	31/12/2016	30/6/2017	22/5/2015

Part B: The Final Report

5. Project Objectives

5.1 Objectives as per original application

1. To continue on the investigation of tackling 2D-CP problems using LFFT algorithm.
2. To extend the LFFT algorithm by taking into consideration of aspect ratios of the bounding boxes
3. To improve on the time-efficiency of the LFFT algorithm.

5.2 Revised objectives

Date of approval from the RGC: Not applicable

5.3 Realisation of the objectives
(Maximum 1 page; please state how and to what extent the project objectives have been achieved; give reasons for under-achievements and outline attempts to overcome problems, if any)

All three objectives have been achieved.

As stated in objectives 1, the main goal of this project is to enhance an algorithm known as LFFT, which is a deterministic 2-D packing algorithm designed for stock-cutting and 2D rectangular cutting and packing (2D-CP) problems with containers of fixed aspect ratios. The main objective of this project is to improve on the solution quality by expanding the solution space into all container shapes and hence all aspect ratios (Objective 2). Here, the main challenge is to find a method to deal with the increased time complexity resulting from the expanded problem space (Objective 3).

To achieve these objectives (and Objective 3 in particular), we have proposed a multi-stage reduction strategy. The new mechanism solves the packing problems in stages, where each stage refines the solution space by examining a successively smaller set of candidate container widths and (Objective 2). Two different approaches were implemented and tested, namely 1) direct extension of LFFT, and 2) a hybrid approach that utilized dynamic programming and a scaled down version of LFFT (Objective 1). Both approaches have been experimentally evaluated using publicly available benchmark problems. The results have been compared with the leading results reported in the literature at the time of writing, namely, the AMHRC results by Bortfeldt, and the DRA and DRA* results published by K. He et al, 2015. From the results, we have shown that both approaches of the extended LFFT mechanism out-performed the currently best-known results in majority (over 63%) of the case tests, with superior overall density and comparable running time. The new results have been published in the proceedings of a peer-review conference (IMECS'16).

5.4 Summary of objectives addressed to date

Objectives *(as per 5.1/5.2 above)*	**Addressed** *(please tick)*	**Percentage Achieved** *(please estimate)*
1. To continue on the investigation of tackling 2D-CP problems using LFFT algorithm.	✓	100%
2. To extend the LFFT algorithm by taking into consideration of aspect ratios of the bounding boxes	✓	100%
3. To improve on the time-efficiency of the LFFT algorithm.	✓	100%

6. Research Outcome

6.1 Major findings and research outcome
(Maximum 1 page; please make reference to Part C where necessary)

We designed and implemented two methods for 2D rectangular cutting and packing (2D-CP) area minimization problems as follows:

i. Direct extension of LFFT. To implement the first method, an algorithm labelled pure-LFFT which directly extends the original LFFT principle was proposed (see Figure 1 of the attached paper). In summary, pure-LFFT extends the original LFFT algorithm by treating the AM problem as a series of stock-cutting problems (see Figure 3 of the attached paper).

ii. Hybrid approach. In the second method, a hybrid approach labelled DP-LFFT, which is based on problem reduction using dynamic programming, was proposed. Here, the original problem is first analyzed by a dynamic programming algorithm, with the goal of finding a set of promising container width, which is then passed to LFFT for optimization. (See Section III of the attached paper).

iii. In both approaches, a multi-stage reduction strategy is employed. The idea is that, starting from a wide range of possible values of container widths, we incrementally reduce the number of candidate widths according to a multi-stage reduction schedule (see Figure 4 of the attached paper). Each stage would employ a successively more complex (and accurate) algorithm to cut down on the number of candidate widths, so that only the ones that deem most promising are retained. Finally, each of the remaining widths after reduction are treated as stock-cutting problems.

The two proposed methods were evaluated using 33 publicly available 2D-CP problem instances, including nine well-known MCNC and GSRC benchmark problems, as well as 24 more recent RPAMP problem instances from the literature.

We compared our result with three of the leading approaches at the time of writing, namely, the AMHRC results by Bortfeldt, and the DRA and DRA* results published by K. He et al, 2015. The results indicate that both of our proposed methods (LFFT), and well as DRA, performed very well for the MCNC and GSRC problems, with both approaches producing the best results in four instances while AMHRC produced the best result in the remaining one problem only. For the 24 new RPAMP problem instance, our proposed methods turn out to be far superior. Out of all 24 problems, LFFT is able to improve on the currently best-known results in more than half of the problems, while DRA is the second-best by leading in 6 of the problems. The overall packing density of LFFT for all problem instances is also superior, while the execution time is comparable. In particular, DP-LFFT is faster than Pure-LFFT which is as expected

The results are summarized in Appendix 1. Sample packing results are illustrated in Appendix 2. All packing results for the benchmark problems are available for downloading at the following footnote .

The findings have been published in IMECS'16.

6.2 Potential for further development of the research and the proposed course of action *(Maximum half a page)*

The ideas we developed (LFFT and the multi-stage reduction strategy) can be applied in the container loading problems (CLP), which is one important application of 3-d rectangular packing (a.k.a 3-d bin packing). There are many variations, including i) multi-container loading problems, where cargo items are to be packed (loaded) into a minimum number of identical containers, ii) multiple bin-size packing problems, where there are several types of containers available, and iii) the open dimension problems, where we need to pack all cargo items into a single container with one or more variable dimension (which is the natural 3-d extension of the 2D-CP area minimization problem we studied in the current project). Obviously. CLP problems have larger problem spaces than 2-d rectangular packing. Therefore, given the time complexity of the LFFT algorithm ($O(n^5 \log n)$), the running time of LFFT in 3D problems would be a major challenge that need to be resolved. Additionally, CLP problems also have several contraints not found in the 2-d equivalence, namely, weight limits, stacking constraints, and stability (i.e, effect of gravity). These constraints would have to be handled if LFFT is to be applied successfully in CLP.

7. Layman's Summary

(Describe in layman's language the nature, significance and value of the research project, in no more than 200 words)

We enhanced a 2-d rectangular cutting and packing (2D-CP) mechanism called LFFT (Least-Flexibility-First principle with Tightness Evaluation). In a 2D-CP problem, we are given a set of rectangular pieces. The goal is to find the smallest container box for holding all pieces without overlapping. The pieces can rotate if necessary. The dimension (including the width-height aspect ratio) are not fixed. In order to deal with the large problem space and to achieve better time-efficiency, we employed a problem reduction mechanism. The idea is that we divide the mechanism into stages. In each stage, we start with a set of candidate packing width, and apply some increasing more complex (and also more accurate) implementation of the LFFT mechanism to refine on the set of candidate widths, and the best ones are kept. We also studied two methods for the initial rounds of problem reduction, namely one that is based on dynamic programming, and one that is purely based on LFFT. Our proposed methods have been evaluated experimentally using well known benchmark problems and the results have been compared favourably with the currently leading results in the literature. The 2D-CP problem has various applications including material cutting, newspaper editing, and VLSI floorplanning.

Part C: Research Output

8. Peer-Reviewed Journal Publication(s) Arising Directly From This Research Project

(Please attach a copy of the publication and/or the letter of acceptance if not yet submitted in the previous progress report(s). All listed publications must acknowledge RGC's funding support by quoting the specific grant reference.)

The Latest Status of Publications				**Author(s)** *(denote the correspond-ing author with an asterisk*)*	**Title and Journal / Book** *(with the volume, pages and other necessary publishing details specified)*	**Submitted to RGC** *(indicate the year ending of the relevant progress report)*	**Attached to this Report** *(Yes or No)*	**Acknowledged the Support of RGC** *(Yes or No)*	**Accessible from the institutional repository** *(Yes or No)*
Year of Publication	**Year of Acceptance** *(For paper accepted but not yet published)*	**Under Review**	**Under Preparation** *(optional)*						
			✓	Chi-Kong Chan*, Tsz Lun Chan and David Y. L Wu	Rectangular Packing by Multi-Stage Problem Reduction, *The European Journal of Operational Research* (EJOR)	No	No	Yes	No

9. Recognized International Conference(s) In Which Paper(s) Related To This Research Project Was / Were Delivered

(Please attach a copy of each conference abstract)

Month / Year / Place	**Title**	**Conference Name**	**Submitted to RGC** *(indicate the year ending of the relevant progress report)*	**Attached to this Report** *(Yes or No)*	**Acknowledged the Support of RGC** *(Yes or No)*	**Accessible from the institutional repository** *(Yes or No)*
16-18 March 2016 Hong Kong	Multi-stage Pseudo-packing-based Mechanism for Area-minimization Rectangular Packing	International MultiConference of Engineers and Computer Scientists 2016 (IMECS'16)	No	Yes	Yes	No

10. Whether Research Experience And New Knowledge Has Been Transferred / Has Contributed To Teaching And Learning
(Please elaborate)

Yes. The 2D packing problem and the proposed solution can serve as examples in the teaching of algorithm design and computational complexity.

11. Student(s) Trained
(Please attach a copy of the title page of the thesis)

Name	Degree Registered for	Date of Registration	Date of Thesis Submission / Graduation
NA			

12. Other Impact
(e.g. award of patents or prizes, collaboration with other research institutions, technology transfer, teaching enhancement, etc.)

NA

13. Public Access Of Completion Report
(Please specify the information, if any, that cannot be provided for public access and give the reasons.)

Information that Cannot Be Provided for Public Access	Reasons
NA	

Note: Principal Investigators of projects approved in 2010/2011 onwards are required to release the completion reports to the public through the RGC website. Completion reports containing information such as abstracts in non-technical terms, objectives, research output including the list of conference papers / publications / journals and research findings and contact information of PIs should be open to public access.

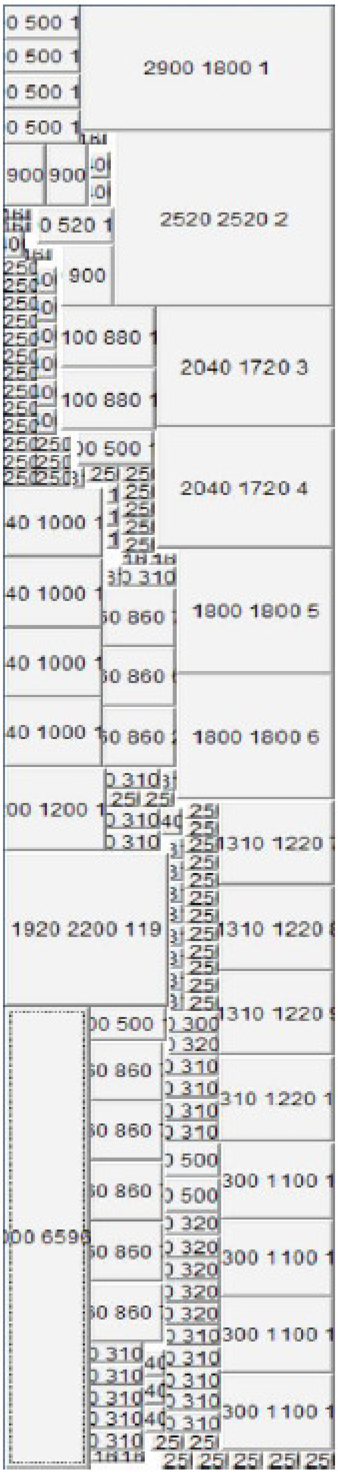
Appendix 1: Final evaluation results

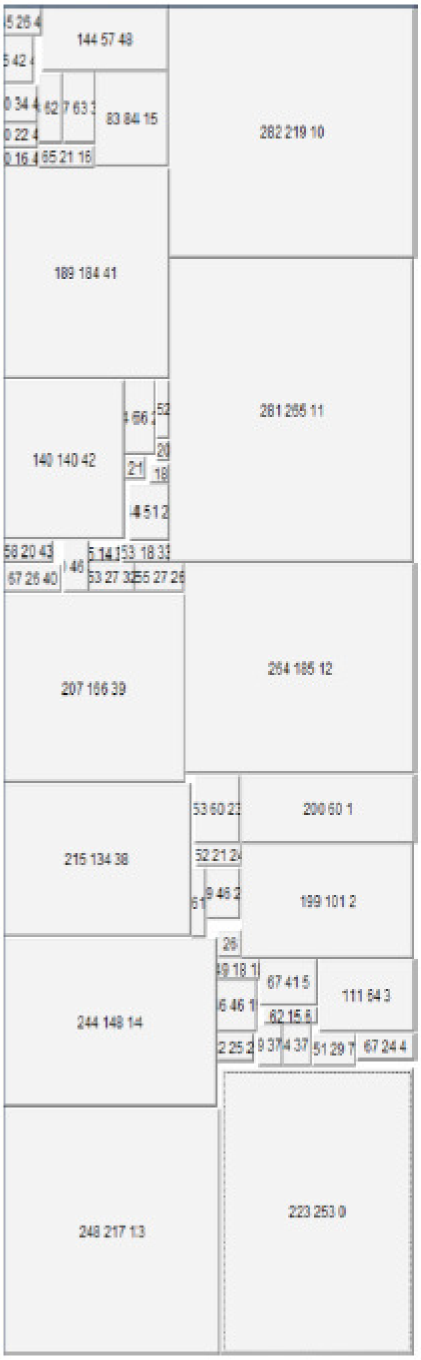
Benchmarks	Instance Name	No.	AMRHC (Bortfeldt 2013) fr(%)	t(s)	DRA (Kun He 2015) fr(%)	t(s)	DRA* (Kun He 2015) fr(%)	t(s)	Pure LFFT (proposed method #1) fr(%)	t(s)	DP-LFFT (proposed method #2) fr(%)	t(s)
MCNC	Ami33	33	**99.01**	116.00	98.77	198.80	98.46	799.23	98.45	2300.00	98.06	2255.00
	Ami49	49	98.58	1752.00	**98.58**	986.46	97.72	3636.98	98.27	1391.00	98.27	1391.00
GSRC	N100	100	98.72	1915.00	98.82	628.03	98.40	319.73	**98.98**	3777.00	**98.98**	2230.00
	N200	200	99.09	37.00	99.51	577.40	99.13	796.05	**99.60**	4076.00	**99.60**	1826.00
	N300	300	99.03	39.00	**99.61**	44.53	99.21	37.10	99.29	3856.00	99.10	567.00
RPAMP (Imahori et al. 2015)	Rp100	100	99.06	59.00	99.13	348.10	98.83	424.97	**99.13**	2724.00	**99.13**	2226.00
	Pcb146	146	98.85	2250.00	99.01	1130.26	98.52	4525.28	**99.08**	5870.00	**99.08**	3485.00
	Rp200	200	99.11	14.00	**99.53**	1624.70	98.73	4958.69	99.37	9060.00	99.28	4788.00
	Pcb500	500	99.07	554.00	**99.41**	337.69	99.21	262.86	99.34	5518.00	99.22	3687.00
RPAMP 50 (Bortfeldt 2013)	1	50	98.65	2501	99.19	1027	98.33	2780	**99.42**	1510	**99.43**	1141
	2	50	97.74	1553	98.50	1279	98.26	3280	**98.91**	6340	**98.89**	2932
	3	50	98.60	1697	98.55	1161	98.20	2123	**99.24**	3946	**99.12**	1864
	4	50	**99.70**	1335	99.56	970	99.56	1335	99.60	2637	99.60	1056
	5	50	99.27	1420	**99.48**	1194	99.48	1528	99.48	4572	99.12	1496
	6	50	99.51	1410	**99.51**	892	99.51	1148	99.39	2915	99.39	961
	7	50	98.73	2972	98.03	1225	98.48	3083	**99.37**	1575	**99.37**	1190
	8	50	97.79	1556	97.12	1460	96.08	4835	**98.87**	7069	**98.87**	5211
	9	50	98.51	1785	97.73	1421	97.82	3442	**99.07**	2613	**99.14**	2262
	10	50	99.64	1362	99.51	945	**99.71**	1258	99.65	1858	99.65	752
	11	50	99.19	1410	99.02	1271	98.97	2306	**99.33**	3614	**99.33**	2026
	12	50	99.56	1434	99.50	1123	**99.70**	1625	99.19	3669	99.19	1471
RPAMP 200 (Bortfeldt 2013)	13	200	99.44	2936	**99.74**	1377	99.20	6767	99.55	2591	99.55	1681
	14	200	99.26	4019	99.42	2961	99.30	3602	**99.62**	6452	**99.64**	2534
	15	200	99.39	3771	**99.63**	1759	99.19	6312	99.62	3562	99.62	1655
	16	200	99.23	2321	**99.73**	1169	98.89	5774	99.37	1626	99.369	817
	17	200	99.20	1507	98.61	1673	98.00	7991	**99.31**	6198	**99.34**	1668
	18	200	99.01	1731	99.41	1047	98.84	6563	**99.59**	4007	**99.49**	1632
	19	200	99.61	9948	**99.80**	1030	99.41	7347	99.74	2967	99.67	1963
	20	200	99.13	3363	98.62	2225	98.30	5136	**99.56**	9536	**99.56**	4968
	21	200	99.50	6151	99.51	3438	99.40	3997	**99.73**	3856	**99.73**	2224
	22	200	99.47	2300	99.63	1557	99.31	1725	**99.66**	2762	99.52	1591
	23	200	98.75	1755	98.82	837	98.67	3481	98.27	8252	**99.36**	1680
	24	200	98.72	1886	99.50	877	98.89	3757	**99.68**	3262	**99.68**	1171
Average			99.03		99.11		98.78		**99.26**		**99.28**	

Overall result

- LFFT leads in 18 benchmark problems.
- DRA leads in 10 benchmark problems.
- DRA* leads in 2 benchmark problems.
- AMRHC leads in 1 benchmark problem.

Appendix 2: Sample packing outputs: pcb146 (top left), Bortfeldt- 50-h07 (top right), Bortfeldt-50-h01 (Bottom)





Multi-stage Pseudo-packing-based Mechanism for Area-minimization Rectangular Packing

Chi-Kong Chan, Dongyang Wu and Yu-Liang Wu

Abstract— **LFFT is a proven heuristics-based iterated search mechanism for 2D rectangular packing problems, and, in particular, for bounding box packing and stock cutting problems. It had also been applied to area-minimization problems in the past, and while the preliminary results were promising, they were still suboptimal because of the problem space complexity. Recent advances in 2D packing points out a new direction, namely, a reduction-based approach that uses dynamic programming to reduce a problem into a more manageable set of stock cutting problem instances. Inspired by the recent advances, we introduce in this paper a multi-stage reduction mechanism for handling the area minimization problem using dynamic programming and an extended version of LFFT. The results for benchmark problems are good as the new approach is able to improve on the state-of-the-art results in majority of the test cases.**

Index Terms— **area minimization, two dimensional packing problems**

I. Introduction

Two-dimensional rectangular cutting and packing (2D-CP) problems are well-known problems where rectangular pieces need to be placed in a single container box without overlapping. Depending on the problem requirement, the goal is to either maximize the total packed area given a fixed size container (called the bounding box packing problem), or, alternatively, to minimize the size of the container box for holding all pieces (called the area minimization problem (AM)[1]). The problem has practical applications in a number of manufacturing and job allocation problems, for instance, in VLSI floor planning problems and in metal or paper cutting. A variation of the area minimization problem is called the stock-cutting problem (SC), where all rectangles need to be packed into a container of fixed width, with the objective of minimizing the height of the container while packing all rectangles. This paper focuses mainly on AM problems but contains references to SC.

2D-CP problems are NP-complete problems (one can easily realize this by noting that both AM and SC are two-dimensional extension of the well-known NP-complete problem of one-dimensional bin packing). Because of this, most proposed solutions focused on finding approximate solutions using various combinations of heuristic and meta-heuristics approaches [1][5]. Two of the well-known and still commonly used heuristics are the Bottom-Left (BL) and Bottom-Left-Fill (BLF) heuristics, with time complexities of $O(n \log n)$ and $O(n^3)$ respectively . In practice, BL and BLF are rarely used on their own. Instead, they are either coupled with meta-heuristics algorithms (e.g., genetic algorithm or simulated annealing), or that they are embedded into higher-level search mechanisms. Such examples can be found in both SC (e.g., [8][9]) and AM (e.g., [10]) problems.

In recent years, a number of works based on advanced search mechanisms coupled with various heuristics have appeared. Our current work is the result of the confluence of two different lines of approaches proposed in the past decade. The first one is based on an idea which we label here as the *pseudo-packing-based* approaches. This approach can be traced back to the *Least-Flexibility-First* (LFF) principle first proposed in [2], which was later enhanced as the LFFT algorithm in [3]. LFF and LFFT are deterministic 2D packing algorithms originally designed for bounding-box packing problems, but can be adapted for other types of packing as well. The pseudo-packing-based mechanism is a tree-based search mechanism that places each rectangle temporarily on each candidate location (called a *move*) in turn for evaluation. Each move is evaluated iteratively using a fast evaluation heuristics that pseudo-packs the remaining rectangles for evaluation purpose. The evaluation heuristics employed by LFF and LFFT are the well-known BL heuristics and a new *tightness* heuristics respectively. This approach turned out to be very successful, especially for stock-cutting problems, but also for other types of 2D packing in general, as demonstrated by well-known benchmark problem instances.

Since then, this idea of utilizing a pseudo-packing-based mechanism in combination with a fast heuristics for iterated greedy evaluation appeared in a number of subsequence works. For example, a search mechanism called A_1 that is similar in concept to the one adopted by LFFT was employed in [11]. This work differs from LFFT in that they employed a different (and more complex) distance-based heuristic for the iterated greedy evaluation part. Hence, the packing densities have been improved, but at a trade-off of higher time complexities. Later on, a *Best fit Algorithm* (BFA) was proposed in [13]. While also based on a similar pseudo-packing-based mechanism, this work introduced a new evaluation heuristics called BFA, which calculates the "smooth degree" of candidate packing positions. BFA has been applied to bounding box packing problems.

However, there is a drawback with the pseudo-packing-based approaches when applied to area-minimization problems. That is, the original LFF

Manuscript received Jan 8, 2016; revised 20 Jan, 2016
This work was supported by the Research Grants Council of Hong Kong (Project Ref No. UGC/FDS14/E01/14)
C.-K. Chan is with the Hang Seng Management College, Shatin, Hong Kong (e-mail: chanck@ hsmc.edu.hk).
D. Wu is with the Hang Seng Management College, Shatin, Hong Kong
Y-L Wu was with the Chinese University of Hong Kong. The is currently serving as the CEO of Easy-Logic Technology Ltd.

[1] Also called the RPAMP problem

ISBN: 978-988-14047-6-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

principle, including the pseudo-packing-based mechanism which was adopted by the subsequence works, was originally designed for the bounding box packing problem. To adapt it for area-minimization, one needs to repeatedly restart the mechanism using containers of increasing sizes, until all pieces can be packed. In the original LFF and LFFT approaches (and also [11]), for instance, because of time-efficiency considerations, this was simply done by attempting *squares* containers of increasing sizes. Needless to say, this approach was sub-optimal; as candidate solutions involving containers with aspect ratios other than the default 1:1 squares were simply not considered (the aspect ratio is the respective ratio of a container's length and width).

The solution to this problem seems to be found in the second line of approaches, namely the *reduction-based* approaches [5][6]. This line of research, which appeared more recently, makes use of two-stage algorithms that first locate a promising set of aspect ratios (or container width values), and then computes the best packing results accordingly using various heuristics. A good example is the AMRHC mechanism [6]. This work divides the AM problem into two smaller sub-problems. First, a 2D-knapsack problem (KP) for determining a promising set of container widths is solved. Then, for each candidate width, a packing solution is found by treating it as a stock-cutting problem. A family of algorithms (more precisely, various combinations of algorithms for KP and SC) were studied in [6]. In practice, however, a combination that uses dynamic programming for KP and a heuristics called BFDH* for stock-cutting was reported to be the most effective. Like [11], this approach can achieve higher packing density than LFFT because it can extend the problem search space to other aspect ratios. Note that, however, this was also achieved at a price of increased time complexity.[2] Very recently, an approach named DRA was proposed in [15]. Like AMRHC, DRA is also a two-stage reduction approach, but it employs an extended version of BFA for stock cutting evaluation. However, DRA also has a large time complexity.[3]

Our current paper is inspired by these recent advances. Our approach is a multi-stage reduction mechanism utilizing an iterated pseudo-packing-based procedure. A dynamic programming process is first performed to produce a set of promising candidate widths, which are then subjected to one or more rounds of packing or further evaluation using LFFT, treating each case as independent stock-cutting problem. Note that DP and LFFT are selected because of their demonstrated performance for the respective sub-problems.

The remaining of the paper is organized as follows. In Section two we define the 2-D packing problems, and present the basic LFFT algorithm. Section three proposes an extension of LFFT for area-minimization problems using DP and a multi-stage extension of LFFT. Section four presents the experiment results. Section five concludes.

II. 2D-CP Problem and the LFFT Principle

A. Problem description

The 2-D rectangular packing area-minimization problem (AM) can be stated as follows. Given a set of n rectangular pieces, and a bounding container box b, we need to place all pieces into b without overlapping, with the objective of minimizing the area of b. In this work, we assume that the aspect ratio of the container b (*i.e.*, `width`(b) / `length`(b)) is not fixed. The rectangles can be rotated by 90 degrees if necessary.

Another problem that is closely related to AM is the two-dimensional stock cutting (SC) problem [4][8][9]. Again, given n rectangular pieces, we need to pack the rectangles into a strip of material of fixed base-line width without overlapping. The goal here is to minimize the height of the strip used to pack all pieces.

B. The LFFT Principle

The *Least Flexibility First Principle with Tightness Evaluation* (LFFT) is a 2D-CP mechanism. The basic principle is to pack the least flexible rectangles, which are the longer ones, into one of the least flexible packing locations, which are the corners. Here, a corner can be formed by any packed rectangles or the sides of the container box. The LFFT principle is realized using a pseudo-packing-based algorithm (LFFT-PsP) which is illustrated in Figure 1. The rectangles are packed in steps. In each step, the q longest remaining rectangles are evaluated at each of the corners in turns (line 4). Here, q is a parameter for controlling the size of the search-space at each step: large q should produce better solutions but at a trade-off of longer execution time. (This feature will be useful in the multi-stage strategy for AM problems, described in the next section). A to-be-packed rectangle and a fitting corner constitutes a move (line 3). Each move m is *pseudo-packed* in turn (line 5), and then evaluated using an *iterated greedy evaluation procedure* (LFFT-IGE) as follows (lines 6, 10-16): in each iteration of LFFT-IGE, we first re-compute an updated list of remaining moves (line 11), and each move m' is evaluated using a fast evaluation heuristic (line 12) and the best one is pseudo-packed. The fast evaluation heuristic chosen for LFFT is a *tightness* heuristic which will be described in the next subsection. The iterated evaluation procedure repeats until no more moves is possible. The resulting packing density then serves as the score for the original move m that was being evaluated (line 15), and the score is passed back to LFFT-PsP (line 6). These steps repeat until all moves belonging to the longest q rectangles have been evaluated, and the best move is selected. The algorithm repeats until no more moves is possible.

Note that the *iterated greedy evaluation procedure* (LFFT-IGE) can also function as a 2D-CP packing mechanism on its own. For example, it can be applied directly for very large problems where the LFFT-PsP algorithm is not suitable, or it can be used as a procedure for evaluating candidate packing moves as described above.

C. A Tightness heuristics for fast evaluation

LFFT attempts to pack a rectangle into the best fitting corners. This idea is approximated using an eight points *tightness* heuristic, which is illustrated in Figure 2. For each candidate move, we look at the points that are immediately adjacent to

[2] As noted in [6], their running time for the benchmark problem sets exceeded those of LFFT despite using a faster machine.

[3] The worst case time complexity of DRA is O(n^{10}), according to [15].

ISBN: 978-988-14047-6-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

***Pseudo-packing-based Packing Algorithm* (LFFT-PsP)**
Input: 1. A set of rectangles. 2. The parameter q.
1. **Repeat** the following steps (2 – 9) **until** all rectangles are successfully packed, or that there are no more legal moves.
2. Let L be the list of the remaining longest q rectangles that are not yet packed.
3. Generate an updated list M of (*rectangle, corner*) pairs, where *rectangle* is in L, and *corner* is a corner location that the current rectangle can be placed without overlapping. Each (*rectangle, corner*) pair constitutes a *move*.
4. **For each** move m=(r, c) in M
5. Pseudo-pack m by temporarily placing rectangle r at the corner c
6. Evaluate move m using the *Iterated Greedy Evaluation Procedure* (LFFT-IGE), and use the resulting packing density as the score of m.
7. Undo move m
8. **End for-each**
9. Select the move in M with the highest score and pack it permanently. Update L and M.

***Iterated Greedy Evaluation Procedure* (LFFT-IGE)**
10. **Repeat** until all remaining rectangles have been pseudo-packed, or that no more move is possible
11. Let M' be the updated list of remaining moves.
12. For each move m' in M', evaluate its fitness using a fast evaluation heuristics (e.g., the *tightness* heuristics).
13. Select the move in $M'$with the highest fitness values. Pseudo-pack this move.
14. **End-Repeat**
15. Compute the resulting packing density
16. Undo all pseudo-packed rectangles in steps 10-14.

Fig. 1. The LFFT Pseudo-Packing-based Packing Algorithm and the Iterated Greedy Evaluation Procedure

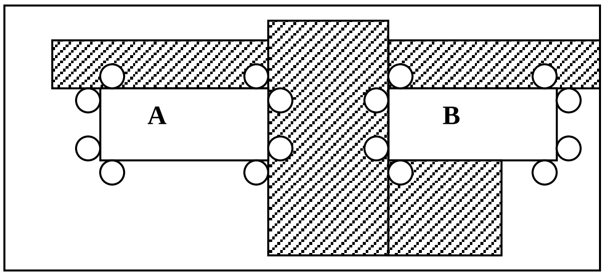


Fig. 2. The *Tightness* heuristics

***LFFT procedure for Stock-cutting* (LFFT-SC)**

Input: 1. A set of rectangles. 2. Container width w.
Initialization:
1. Let *rect_area* be the sum of the areas of all rectangles
2. Let h = $\lceil rect\_area \ / \ w \rceil$

Procedure:
3. **Repeat until** all rectangles are successfully packed.
4. Execute LFFT-PsP *or* LFFT-IGE to pack the rectangles, with container width w and height h.
5. **If** all rectangles can be packed successfully
6. **Quit**.
7. **else**
8. Let h = $\lceil h \times (1 + \delta) \rceil$

Fig. 3. Procedure for adapting LFFT for stock cutting

each corner of the candidate rectangle. That is, suppose a rectangle has the corners $\{(x_0, y_0), (x_0, y_1), (x_1, y_1), (x_1, y_0)\}$, we check whether the 8 corner adjacent points $\{(x_0, y_0 - \delta), (x_0 - \delta, y_0), (x_0 - \delta, y_1), (x_0, y_1 + \delta)$ $(x_1, y_1 + \delta)$, $(x_1 + \delta, y_1)$, $(x_1 + \delta, y_0),)$, $(x_1, y_0 - \delta),)\}$ are occupied. The fitness of each move is then defined as the number of corner adjacent points that is not located in open space (that is, not in packed area, and not exceeding the container boundary). In Figure 2, the fitness of moves A and B are four and five respectively.

D. LFFT for Stock Cutting (LFFT-SC)

LFFT can be adapted quite easily for stock-cutting problems. The idea is illustrated in Figure 3 as the LFFT-SC mechanism. We start with a minimal container with base width w. The LFFT-PsP algorithm is applied repeatedly with increasing container height, until a packing solution is found. Alternatively, for very large problem instances, the faster LFFT-IGE procedure can be used to replace LFFT-PsP (step 4 of Figure 3).

E. Computational Complexity of LFFT

The placement of a rectangle in each step of the algorithm will occupy one or more corners, and, at the same time, also generates a few new corners. Therefore, the number of corners at any time should be proportional to n, where n is the total number of rectangles, and the length of the move list in each step is bounded by O(q * n), where q is the parameter as mentioned above. In our implementation, the packed rectangles are stored using a k-d tree data structure [14], which provides a fast O(log n) region search operations. As a result, the complexities of LFFT-IGE and LFFT-PsP are O(qn^2log n), and O(qn^4log n) respectively.

III. Multi-stage reduction Strategy for area-minimization problems

A. Multi-stage reduction Strategy

LFFT, with its *Pseudo-packing-based* algorithm, was originally a mechanism designed for the bounding-box packing problem. To adapt it for area minimization problems, where the problem spaces are much larger, we can utilize a multi-stage reduction approach, explained as follows. The idea is that, starting from a wide range of possible values of container widths, we incrementally reduce the number of candidate widths according to a multi-stage reduction schedule. Each stage would employ a successively more complex (and accurate) algorithm to cut down on the number of candidate widths, so that only the ones that deem most promising are retained. Finally, each of the remaining widths

	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5
n<100	**DP** (p_{max}=10) **Output:** best 200 widths	**LFFT-SC** with **LFFT-IGE** **Output:** best 50 widths	**LFFT-SC** with **LFFT-PsP** (q=1) **Output:** best 20 widths	**LFFT-SC**with **LFFT-PsP** (q=5) **Output:** best 5 widths	**LFFT-SC** with **LFFT-PsP** (q=10) Output: best packing result
100 ≤ *n*<200	**DP** (p_{max}=20) **Output:** best 100 widths	**LFFT-SC** with **LFFT-IGE** **Output:** best 20 widths	**LFFT-SC** with **LFFT-PsP** (q=1) **Output:** best 3widths	**LFFT-SC** with **LFFT-PsP** (q=5) Output: best packing result	
200 ≤ *n*<300	**DP** (p_{max}=40) **Output:** best 80 widths	**LFFT-SC** with **LFFT-IGE** **Output:** best 3 widths	**LFFT-SC** with **LFFT-PsP** (q=1) Output: best packing result		
300 ≤ *n*≤ 500	**DP** (p_{max}=60) **Output:** best 50 widths	**LFFT-SC** with **LFFT-IGE** **Output:** best packing results			

Fig. 4. A multiple-stage reduction schedule for area-minimization problems

after reduction are packed using a stock-cutting mechanism (e.g., LFFT-SC) and the best solution is selected.

An example reduction schedule is depicted in Figure 4. In this example, the schedule to be followed is determined by the number of rectangles (n). In all cases, the first stage utilizes a dynamic programming (DP) algorithm for suggesting a promising set of candidate widths (discussed in the next subsection), which are then passed to the next stage. Depending on n, the next few stages either employ LFFT-SC with LFFT-PsP, or LFFT-SC with the less complex LFFT-IGE algorithm. Different values for the parameter q are used in the various stages to keep the running time manageable. In each case, the rectangles are trial-packed using containers of each suggested width in turns, and the widths that produce the best results are kept. For example, given a problem with 150 rectangles. We first employ the DP procedure to produce a list of the 100 most promising width values (stage 1). For each of these widths, we employ LFFT-SC with LFFT-IGE to pack the rectangles. The 20 width values that produce the best results are passed to stage 3, which performs 20 rounds of packing using LFFT-SC with LFFT-PsP and with q=1. The best 3 results are then re-analyzed using LFFT-SC with LFFT-PsP, but this time using q=5 (which produces better results but the expected running time is 5 times as long than the previous stage for each problem instance). The best packing result is then reported. Note that this schedule has been tuned with a goal that the whole packing process can be completed in reasonable time using a single contemporary PC computer, after referencing the running time of a number of related approaches reported in the literature for common benchmark problems (see Section IV for details).

B. Dynamic programming method for 2D-CP problem reduction

For all problem size n, the first stage of the proposed reduction schedule relies on a dynamic programming (DP) procedure for reducing the initial set of possible width values into a more manageable set of promising widths for further evaluation. The DP procedure we use is adopted from an approach described by Bortfeldt in [6] for the Interval Subset Sum Problem (ISSP). The problem can be stated as follows. Given a set of rectangles, and a maximum length p_{max}, we want to determine the number of combinations of no more than p_{max} rectangles whose widths add up to various values. More specifically, we want to know the most frequently occurring sum of rectangle widths, for combinations of no more than p_{max} rectangles. The idea is that those frequently occurring sum-of-widths should also serve as good indicators for promising container widths for the area minimization problem. A good explanation for an implementation of a dynamic programming procedure for ISSP problem is provided in [6], so we shall not repeat the details here. For now, it is sufficient to note that the abovementioned DP procedure has a worst case complexity of O($w_{max}n^2$), where w_{max} is the largest possible sum-of-widths and n is the number of rectangles. In our implementation, the DP procedure finishes within one minute in all problem instances.

IV. Experiments

We implemented the LFFT mechanism (LFFT-PsP, LFFT-IGE, LFFT-SC) as well as the DP procedure on a 3.2 GHz PC with 8 gigabytes memory. [4] The proposed method are tested using 33 publicly available 2D-CP problem instances, including nine well-known MCNC and GSRC benchmark problems, as well as 24 more recent RPAMP problem instances from [6]. Note that our focus is on medium to large problem sets with 50 or more rectangles. For this reason, several smaller MCNC and GSRC problems are not included (with the exception of ami33 and ami49, which are included due to their popularity in the 2-D packing literature).

We compared our result with three of the state-of-the-art approaches at the time of writing, namely, the AMHRC results by Bortfeldt [6], and the DRA and DRA* results published recently by K. He et al. [15]. The results are given in Table 1, where the leading approaches are highlighted using bold fonts. We see that both LFFT and DRA performed very well for the MCNC and GSRC problems, with both approaches producing the best results in four instances, while AMHRC produced the best result in the remaining one problem. For the 24 new RPAMP problem instance, LFFT turns out to be far superior. Out of all 24 problems, LFFT is

[4] Inter(R) Core(TM) i5-4460 3.20 GHz with 8.00 gigabytes RAM, running Windows 7.

able to improve on the currently best-known results in 15 cases (or 63% of the problems), while DRA is the second-best by leading in 6 of the problems (25%). The overall packing density of LFFT for all problem instances is also superior (99.3% for LFFT vs 99.1% for DRA vs 98.8% for AMHRC), while the execution time is comparable. Note that LFFT also has a relatively smaller complexity than DRA. [5] The detailed packing results for each problem instance can be viewed at the hyperlink included in following footnote.[6]

V. Conclusion

The 2D rectangular packing area-minimization problems (AM) are harder than traditional 2D packing problems due to their larger problem space. In this paper, we proposed a multi-stage reduction approach for AM problems by extending a mechanism called LFFT, which is a proven mechanism for 2D packing and stock-cutting. A dynamic programming procedure and a scaled-down version of LFFT are first employed according to a reduction schedule, which reduce the problem to a smaller set of more manageable stock cutting problem instances. One or more executions of LFFT then follow for producing the final packing results, while further reducing the number of candidate width values in each execution. This approach has been evaluated using publicly available problem instances and the results are encouraging, with the extended LFFT mechanism out-performing the currently best-known results in 63% of the case tests.

References

[1] Lodi, A., Martello, S., & Vigo, D. (1999). Heuristic and metaheuristic approaches for a class of two-dimensional bin packing problems. *INFORMS Journal on Computing*, 11(4), 345-357.

[2] Wu, Y. L., Huang, W., Lau, S. C., Wong, C. K., & Young, G. H. (2002). An effective quasi-human based heuristic for solving the rectangle packing problem. *European Journal of Operational Research*, 141(2), 341-358.

[3] Wu, Y. L., & Chan, C. K. (2005). On improved least flexibility first heuristics superior for packing and stock cutting problems. In *Stochastic Algorithms: Foundations and Applications* (pp. 70-81). Springer Berlin Heidelberg.

[4] Burke, E. K., Kendall, G., & Whitwell, G. (2004). A new placement heuristic for the orthogonal stock-cutting problem. *Operations Research*, 52(4), 655-671.

[5] Wetweerapong, J., & Remsungnen, T. (2010). Solving the Rectangular Packing Problem by a New Bottom-Left Placement Method Incorporated with a Multi-Start Local Search. *Foundations of Computing and Decision Sciences*, 35, 127-141.

[6] Bortfeldt, A. (2013). A reduction approach for solving the rectangle packing area minimization problem. European Journal of Operational Research, 224(3), 486-496.

[7] Wäscher, G., Haußner, H., & Schumann, H. (2007). An improved typology of cutting and packing problems. *European Journal of Operational Research*,183(3), 1109-1130.

[8] Burke, E. K., Kendall, G., & Whitwell, G. (2009). A simulated annealing enhancement of the best-fit heuristic for the orthogonal stock-cutting problem. *INFORMS Journal on Computing*, 21(3), 505-516.

[9] Leung, S. C., Zhang, D., Zhou, C., & Wu, T. (2012). A hybrid simulated annealing metaheuristic algorithm for the two-dimensional knapsack packing problem. *Computers & Operations Research*, 39(1), 64-73.

[10] H. Zhou and J. Wang, ACG- adjacent constraint graph for general floorplans, ICCD 2004 pp 572-575

[11] Chen, M., & Huang, W. (2007). A two-level search algorithm for 2D rectangular packing problem. *Computers & Industrial Engineering*, 53, 123-136.

[12] Li, Y., Li, Y., & Zhou, M. (2011, December). A greedy algorithm for wire length optimization. In Electronics, Circuits and Systems (ICECS), 18th IEEE International Conference on (pp. 366-369), 2011

[13] K. He, W. Huang, and Y. Jin. "An efficient deterministic heuristic for two-dimensional rectangular packing." Computers & Operations Research 39.7 (2012): 1355-1363.

[14] J.L. Bentley, Multidimensional binary search trees used for associative searching, *Communication of ACM* 18 (9) (1975) 509–517.

[15] K. He, P. Ji, and C. Li. "Dynamic reduction heuristics for the rectangle packing area minimization problem." *European Journal of Operational Research* 241.3 (2015): 674-685.

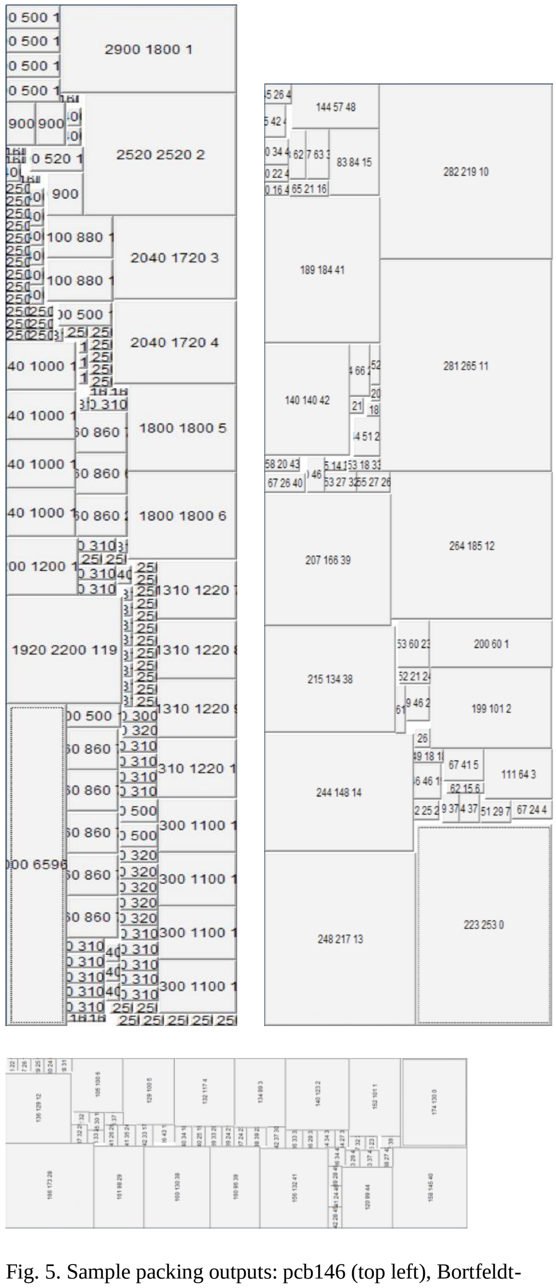


Fig. 5. Sample packing outputs: pcb146 (top left), Bortfeldt-50-h07 (top right), Bortfeldt-50-h01 (Bottom)

[5] The reported complexity for one execution of DRA, as reported in [15] is O(n^{10}). However, one must note that this is a worst case complexity, assuming there can be up to O(n^3) action spaces at any instance. Their average case complexity should be around O(n^8), assuming there are O(n) action spaces at any time. In comparison, the average case complexity for one execution of LFFT is O(n^4 log n), also assuming there are O(n) valid moves at any time.

[6] Available at: https://drive.google.com/a/hsmc.edu.hk/file/d/0B0jDBeaFyEu-QzlFQUkwYlZkOFU/view?pref=2&pli=1

TABLE I
RESULTS OF COMPARISON

Benchmarks	Instance		AMRHC (Bortfeldt) [6]		DRA (He et al.) [15]		DRA* (He et al.) [15]		LFFT	
	Name	*N*	density(%)	t(s)	density(%)	t(s)	density(%)	t(s)	density(%)	t(s)
Well known Benchmarks (MCNC & GSRC)	Ami33	33	**99.01**	116.00	98.77	198.80	98.46	799.23	98.056	2255
	Ami49	49	98.58	1752.00	**98.58**	986.46	97.72	3636.98	98.266	1391
	N100	100	98.72	1915.00	98.82	628.03	98.40	319.73	**98.980**	2230
	N200	200	99.09	37.00	99.51	577.40	99.13	796.05	**99.597**	1826
	N300	300	99.03	39.00	**99.61**	44.53	99.21	37.10	99.101	567
	Rp100	100	99.06	59.00	99.13	348.10	98.83	424.97	**99.133**	2226
	Pcb146	146	98.85	2250.00	99.01	1130.26	98.52	4525.28	**99.081**	3485
	Rp200	200	99.11	14.00	**99.53**	1624.70	98.73	4958.69	99.275	4788
	Pcb500	500	99.07	554.00	**99.41**	337.69	99.21	262.86	99.219	3687
RPAMP 50 [6]	1	50	98.65	2501	99.19	1027	98.33	2780	**99.427**	1141
	2	50	97.74	1553	98.50	1279	98.26	3280	**98.886**	2932
	3	50	98.60	1697	98.55	1161	98.20	2123	**99.119**	1864
	4	50	**99.70**	1335	99.56	970	99.56	1335	99.598	1056
	5	50	99.27	1420	**99.48**	1194	99.48	1528	99.115	1496
	6	50	99.51	1410	**99.51**	892	99.51	1148	99.390	961
	7	50	98.73	2972	98.03	1225	98.48	3083	**99.372**	1190
	8	50	97.79	1556	97.12	1460	96.08	4835	**98.867**	5211
	9	50	98.51	1785	97.73	1421	97.82	3442	**99.135**	2262
	10	50	99.64	1362	99.51	945	**99.71**	1258	99.650	752
	11	50	99.19	1410	99.02	1271	98.97	2306	**99.332**	2026
	12	50	99.56	1434	99.50	1123	**99.70**	1625	99.193	1471
RPAMP 200 [6]	13	200	99.44	2936	**99.74**	1377	99.20	6767	99.552	1681
	14	200	99.26	4019	99.42	2961	99.30	3602	**99.644**	2534
	15	200	99.39	3771	**99.63**	1759	99.19	6312	99.620	1655
	16	200	99.23	2321	**99.73**	1169	98.89	5774	99.369	817
	17	200	99.20	1507	98.61	1673	98.00	7991	**99.337**	1668
	18	200	99.01	1731	99.41	1047	98.84	6563	**99.494**	1632
	19	200	99.61	9948	**99.80**	1030	99.41	7347	99.666	1963
	20	200	99.13	3363	98.62	2225	98.30	5136	**99.560**	4968
	21	200	99.50	6151	99.51	3438	99.40	3997	**99.733**	2224
	22	200	99.47	2300	99.63	1557	99.31	1725	99.521	1591
	23	200	98.75	1755	98.82	837	98.67	3481	**99.364**	1680
	24	200	98.72	1886	99.50	877	98.89	3757	**99.678**	1171
Average			99.03		99.11		98.78		**99.283**	

ISBN: 978-988-14047-6-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)